Chapter 14

A Two-Level Decision Problem

This chapter

This chapter studies a two-level decision problem. There is a government at the policy making level wanting to influence the behavior of several companies at the policy receiving level. The government intends to attain its objectives by implementing tax and subsidy policies. The individual companies make their own optimal decisions *given* the tax and subsidy rates announced by the government. A two-level model of the problem is presented, and subsequently solved using two alternative solution approaches. The first is intuitive and general, but requires extensive computations. The second approach is theoretical and less general, but is computationally efficient. Both approaches are described in detail, and can easily be implemented in Aimms. A data set and some basic data manipulations are included for illustrative purposes.

Reference

A two-level program similar to the one discussed in this chapter can be found in [Bi82].

Keywords

Nonlinear Program, Auxiliary Model, Customized Algorithm, Mathematical Derivation, What-If Analysis, Worked Example.

14.1 Problem description

Waste water

Consider a set of manufacturing companies situated on a river, producing similar products. The demand for these products is such that each company operates at maximum capacity for maximum profitability. Unfortunately, these manufacturing plants all produce waste water, which is dumped into the river. This practice has caused the water quality to deteriorate.

Waste treatment

Through public and governmental pressure all the companies have installed waste treatment plants. These plants use filters for waste removal. The exact amount of waste removed is controlled by varying the number of filters and their cleaning frequency. The cost of removing waste from waste water becomes increasingly expensive as the waste is more dilute.

Government monitoring

The government monitors both the production of waste water and the amount of waste removed at each manufacturing site. The resulting data forms the basis of ecological water studies from which recommendations for maintaining river water quality are made.

Taxes and subsidies

The government has various options to control the amount of waste dumped into the river. One option is to specify an annual quota for each manufacturer. However, this requires extensive negotiations with each manufacturer and may lead to court actions and resistance. A second and more workable option is to introduce a tax incentive scheme using its legislative power. The idea is to introduce a tax on waste and a corresponding subsidy on waste removal such that the total tax revenue covers not only the total subsidy expenditure, but also the cost of monitoring the plants. The effect of such a scheme, assuming a cost minimizing response from the manufacturing companies, is a reduction in the amount of waste dumped into the river.

Two-level decision problem

The above proposed tax incentive scheme results in a two-level decision problem. At the higher *policy making* level, the government must decide a joint tax and subsidy policy to steer the waste removal decisions by the manufacturing companies at the lower *policy receiving* level. At this lower level, decisions are only made on the basis of cost minimization and are independent of any government objectives.

Different knowledge levels

It is assumed that the government has complete knowledge of the problem, while the manufacturing companies have only limited knowledge. The government has access to all waste production and removal data, and has insight into the cost associated with operating waste treatment plants. The individual manufacturing companies operate independently. They have no insight into the manufacturing operations of their competitors, nor are they aware of the precise objectives of the government.

Example

Consider an example consisting of four manufacturing companies with Table 14.1 representing the total amount of waste produced, the waste concentration per unit of waste water, and an operating efficiency coefficient for each waste treatment plant.

	waste water $[10^3 m^3]$	waste concentration $[\text{kg}/m^3]$	efficiency coef. $[\$\cdot\text{kg}/m^6]$
Company 1	2,000	1.50	1.0
Company 2	2,500	1.00	0.8
Company 3	1,500	2.50	1.3
Company 4	3,000	2.00	1.0

Table 14.1: Waste production by the manufacturing companies

If none of the companies removed waste, then the total annual amount of waste dumped is 15,250 [10^3 kg]. It will be assumed that the government has set its target level to 11,000 [10^3 kg], and that the total monitoring cost is estimated to be 1,000 [10^3 \$].

14.2 Model formulation

Verbal Model

Following is a verbal model statement of the problem. The two-level relationship is reflected by the fact that the policy receiving submodels are nothing more than constraints in the policy making model.

Goal: *the government wants to find a tax and subsidy rate*
Subject to:
- *total revenue from taxation equals total subsidy expenditure plus an amount covering the waste monitoring activities,*
- *the total amount of waste dumped by all companies is less than or equal to a fixed target level, and*
- *for each company the amount of waste dumped by that company is the result of individual cost minimization reflecting waste removal cost, taxes and subsidies.*

In the above formulation, the government decides on the tax and subsidy rates, while the individual companies decide on the level of waste removal *given* the government decision. As stated previously, these companies are not aware of any of the financial or environmental targets set by the government.

Notation . . .

The verbal model statement of the two-level waste removal problem can be specified as a mathematical model using the following notation.

. . . at the policy making level

Parameters:

L	*target level of waste to be dumped annually [10^3 kg]*
K	*total annual cost of government monitoring [10^3 \$]*

Variables:

T	*tax rate for waste produced [\$/kg]*
S	*subsidy rate for waste removed [\$/kg]*

. . . at the policy receiving level

Index:

j	*manufacturing companies*

Parameters:

d_j	*waste concentration observed at j [kg/m^3]*
q_j	*waste water produced annually by j [$10^3 m^3$]*
c_j	*efficiency coefficient of company j [\$ · kg/m^6]*

Variable:

x_j *removal of waste water by j [kg/m^3]*

Tax-subsidy balance

The tax-subsidy balance states that total annual government receipts from taxation must equal the total cost of government monitoring plus its total subsidy expenditure.

$$T \sum_j q_j(d_j - x_j) = K + S \sum_j q_j x_j$$

Total waste limitation

The total waste limitation constraint requires that the annual amount of waste dumped into the river is less than or equal to the target level.

$$\sum_j q_j(d_j - x_j) \leq L$$

Policy receiving submodels

The cost minimization model for each company j concerns the choice of x_j given T and S. The objective function can be written as

$$\underset{0 \leq x_j \leq d_j | T,S}{\textbf{Minimize:}} \quad q_j \left[\left(\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j} \right) + T(d_j - x_j) - S x_j \right] \qquad \forall j$$

Waste removal cost term

The term $\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j}$ denotes the cost associated with the removal of waste from each unit of waste water by manufacturing company j and it is non-linear. The functional form of this term is based on the filtering technology used in the waste treatment plants. Filtering cost becomes essentially infinite if all waste has to be removed from waste water. The coefficient c_j reflects the operating efficiency of each waste treatment plant, and its numeric value is based on historic data.

Tax incentive cost terms

The term $T(d_j - x_j)$ denotes the company's tax expenditure associated with the left-over concentration of waste. The term Sx_j denotes the subsidy for waste removal per unit of waste water.

Strict convexity

Note that the policy receiving models are unconstrained minimization models. The cost functions are strictly convex for all values of S and T. This implies the existence of a unique minimum and thus a unique response from the companies to the government. The curves in Figure 14.1 represent convex function contours, and illustrate the total cost of manufacturing company 1 as a function of the waste removal variable x_1 for several combined values of T and S.

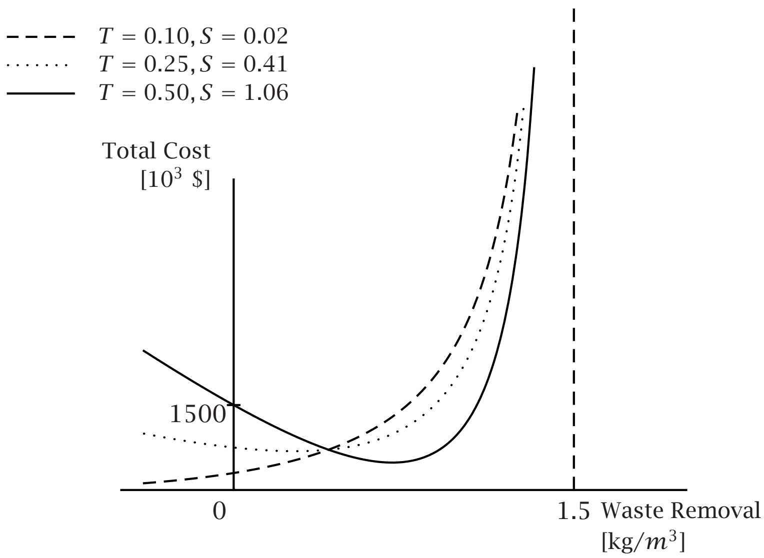

Figure 14.1: Cost function for company 1 for several T and S combinations

Model summary

The following mathematical statement summarizes the model.

Find: T, S

Subject to:

$$T \sum_j q_j (d_j - x_j) = K + S \sum_j q_j x_j$$

$$\sum_j q_j (d_j - x_j) \leq L$$

$$\underset{0 \leq x_j \leq d_j | T,S}{\textbf{Minimize:}} \quad q_j \left[\left(\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j} \right) + T(d_j - x_j) - S x_j \right] \qquad \forall j$$

14.3 Algorithmic approach

This section

This section describes an algorithm to compute the government tax and subsidy rates. The algorithm is an iterative scheme in which the government tax rate is adjusted each iteration. The process continues until the response from all companies is such that the total amount of annual waste dumped roughly equals the target level of total waste. It is straightforward to implement this algorithm in the AIMMS modeling language.

Waste equality

The government realizes that any form of waste removal is costly, and that extra removal cost negatively impacts on the competitive position of the individual companies. Therefore, the waste limitation inequality (restricting the total amount of waste to be dumped into the river to be at most L) can be viewed as an *equality*.

Derive subsidy from tax

Let $G = \sum_j q_j d_j$ denote the total amount of waste that would be dumped into the river without any removal activities. As indicated in the previous paragraph, L is the exact amount of waste that the government wants to be dumped. Under this assumption the tax-subsidy balance becomes $TL = K + S(G - L)$. Once the government decides on a value of T, the corresponding value of S is then computed using the formula $S = (TL - K)/(G - L)$.

Iterate on tax

The above observations, coupled with the unique cost minimizing response from the manufacturing companies, forms the basis of the following simple computational scheme for the government.

1. Decide on an initial tax rate T.
2. Compute the corresponding subsidy rate S.
3. Announce these rates to the companies at the policy making level.
4. Let them respond with their waste removal decisions.
5. Compute the total amount of waste going to be dumped.
6. If approximately equal to L, then stop.
7. If less than L, then decrease T and go to step 2.
8. If greater than L, then increase T and go to step 2.

Proper response

Note that if the amount of dumped waste is too high, the tax rate should go up. This increases the corresponding subsidy rate, thereby providing the companies with a higher incentive to remove more waste. If the amount of dumped waste is less than L, then the tax rate should be decreased. This lowers the subsidy rate and indirectly also the level of waste removal.

Convergence bisection search

The cost minimizing response from the manufacturing companies is a continuous function of the parameters T and S. This means that a small change in T (and thus in S) will result in a small change in the value of the corresponding removal variables x_j. In addition, the corresponding waste removal cost function value for each manufacturing company increases monotonically with an increase in the value of T. As a result, a straightforward bisection search over the tax rate T will lead to a combined tax and subsidy rate for which the amount of waste to be dumped is approximately equal to L. For these rates, the tax-subsidy balance is satisfied by construction.

Hunt phase

In a bisection search algorithm both a lower bound LB and an upper bound UB on the tax rate T are required. An initial lower bound on T is the value for which the corresponding S becomes exactly zero. An initial upper bound on the value of T is not easily determined which is why there is an initial *hunt phase*. In the hunt phase, both a good upper bound and an improved lower bound are found. The computational procedure (in pseudo code) is summarized next.

```
LB := K / L ;
REPEAT
    T := 2 * LB ;
    S := ( T * L - K ) / ( G - L ) ;
    Solve for x_j given T and S ;
    Compute total waste (to be dumped) ;
    BREAK WHEN total waste < target level ;
    LB := T ;
ENDREPEAT ;
UB := T ;
```

Note that the current value of LB is adjusted upwards each iteration until a proper upper bound UB has been determined. At the end of the hunt phase UB = T and LB = T/2.

Bisection phase

The interval found during the hunt phase is used as the starting interval for the *bisection phase*. In this phase either the lower or upper bound is set equal to the midpoint of the current interval until the final value of T has been determined. A small positive tolerance level TOL is introduced to assist with proper termination of this phase.

```
TOL := 1.0E-3 ;
REPEAT
    T := ( LB + UB ) / 2 ;
    S := ( T * L - K ) / ( G - L ) ;
    Solve for x_j given T and S ;
    Compute total waste (to be dumped) ;
    BREAK WHEN abs(total waste - target level) < TOL ;
    If total waste < target level, then UB := T ;
    If total waste > target level, then LB := T ;
ENDREPEAT ;
```

14.4 Optimal solution

This section

This section reports on the results of solving the two-level decision example using AIMMS and with the above algorithm implemented. In addition to the initial optimal solution, several sensitivity graphs with respect to the target level L are included.

Initial solver failure

When applying the above algorithmic procedure to the example, the cost minimization models need to be solved for given values of T and S. However, during the solution process, the nonlinear solver reported ‘Derivative evaluation error … Division by zero’. This error occurred during the evaluation of the term $(\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j})$ because the solver had made too large a step while increasing the value of x_j, causing the value of x_j to get too close to value of d_j.

Add extra bounds

When building nonlinear programming models, it is a good strategy to bound variables away from values for which the functions are not well-defined. The solver takes these bounds into account at all times during the solution process, thereby avoiding the occurrence of extremely large or small values of the decision variables. For the example, limiting x_j to $d_j - \epsilon$ with $\epsilon = 0.001$ was sufficient.

Subsequent solver failure

After adding the extra bound on x_j, another computational problem occurred. The nonlinear solver reported 'Solver found error ... Jacobian element too large = 1.0E+06'. This error occurred during the evaluation of the derivative of the term $(\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j})$ during the solution process. This type of error often occurs when the underlying model is badly scaled. Under these conditions, the problem can usually be avoided by adjusting the measurement unit associated with parameters, variables and constraints so that the values of these identifiers become comparable in size.

Provide starting solution

However, in the case of the example now under investigation, it turned out that the error was not due to bad scaling. Instead, the derivative of the first term $(\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j})$ grows too fast when x_j approaches d_j. One option is to bound x_j even further away from d_j. Another option is to provide a starting value from which the algorithm does not search for new values near d_j. The last option was tried, and the x_j variables were initialized to $d_j/2$. With these starting values, the nonlinear solver solved the cost minimizing models to optimality, and the iterative procedure to compute the optimal tax value converged properly.

Optimal solution

The iterative procedure to determine the optimal tax rate consisted of 2 steps to complete the hunt phase and 10 steps to complete the bisection phase. During the hunt phase, the initial lower bound LB increased from $K/L = 0.0909$ $(1,000/11,000)$ to 0.1820 with a final upper bound twice that size. The optimal value of T was computed to be about \$0.239 per unit waste with a corresponding tax rate S of \$0.384 per unit waste removed. The solution values are summarized in Table 14.2. The total tax income for the government is 2,632 [10^3\$] and the total subsidy expense is 1,632 [10^3\$].

	x_j [kg/m^3]	$q_j x_j$ [$10^3 kg$]	Cleaning Cost [10^3\$]	Tax - Subsidy [10^3\$]	Total Cost [10^3\$]
Company 1	0.233	466.567	245.552	427.000	672.552
Company 2	–	–	–	598.145	598.145
Company 3	1.056	1,583.582	570.155	−89.702	480.453
Company 4	0.733	2,199.851	868.328	64.557	932.885

Table 14.2: Optimal removal levels and corresponding costs

The effect of target levels . . .

Once a model is operational, it is quite natural to investigate its properties further by running some experiments. For instance, the government might want to investigate the effect of different target levels on the behavior of the individual manufacturing companies. The experiments reported in this section are based on letting the target pollution level L vary from a restricting value of 5,000 [10^3 kg] to the non-restricting value of 17,500 [10^3 kg]. The results are described in the next several paragraphs.

. . . on waste removal

The waste removal curves in Figure 14.2 indicate that under the tax incentive scheme of the government, the largest waste producer will remove most of the waste. Company 2 is the second largest manufacturing company, but with the cleanest overall production process. The tax incentive program is no longer applicable for this company once the target level L set by the government is 10,000 [10^3 kg] or more. Note that the order in which companies no longer remove waste is the same as the order of waste concentrations d_j.

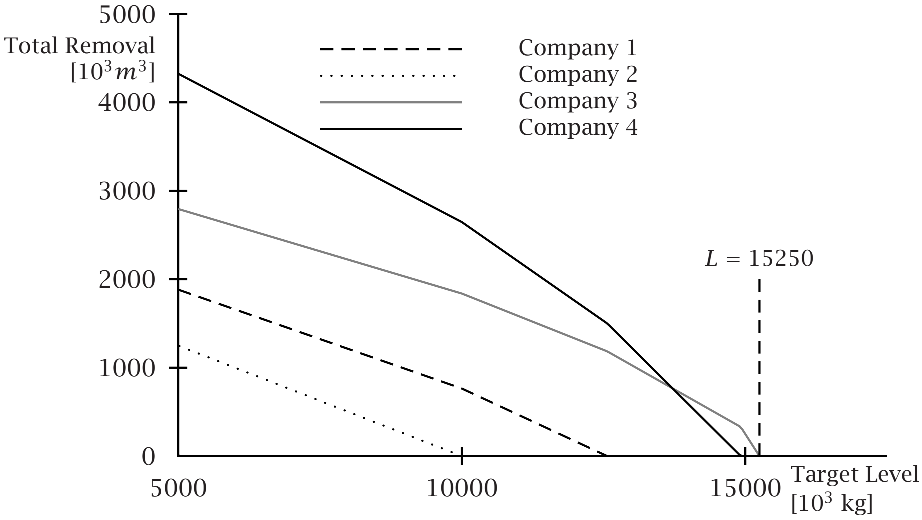

Figure 14.2: The amount of waste removed ($q_j x_j$) per company

. . . on total cost per unit waste

Figure 14.3 shows that the total cost curve for each manufacturing company decreases as the value of L increases. This is to be expected as both the removal cost and the tax cost (due to a lower tax rate) will decrease as L increases. It is interesting to note that the companies with the lowest total waste (i.e. companies 1 and 2) have the highest total cost per unit waste when the target level L is low, and have the lowest total cost per unit waste when the target level L is no longer restricting.

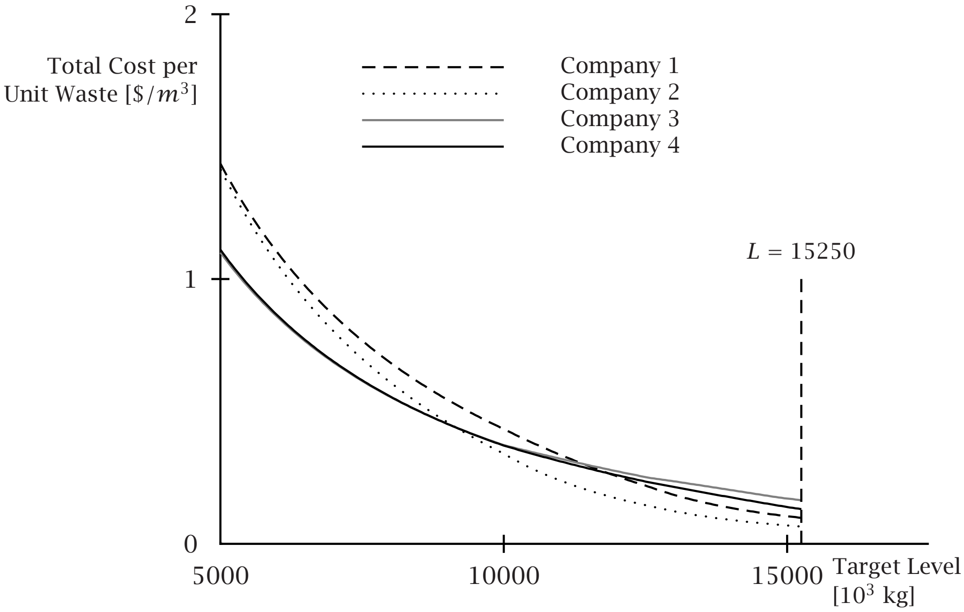

Figure 14.3: The total cost per unit waste
(i.e. $\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j} + T(d_j - x_j) - Sx_j$)

The two curves in Figure 14.4 compare the tax and subsidy rates as a function of the target level *L*. Note that the tax rate is the highest when the target level is the lowest. This is necessary to induce the companies to remove a significant amount of their waste in order to reach that target level.

… on tax & subsidy rates

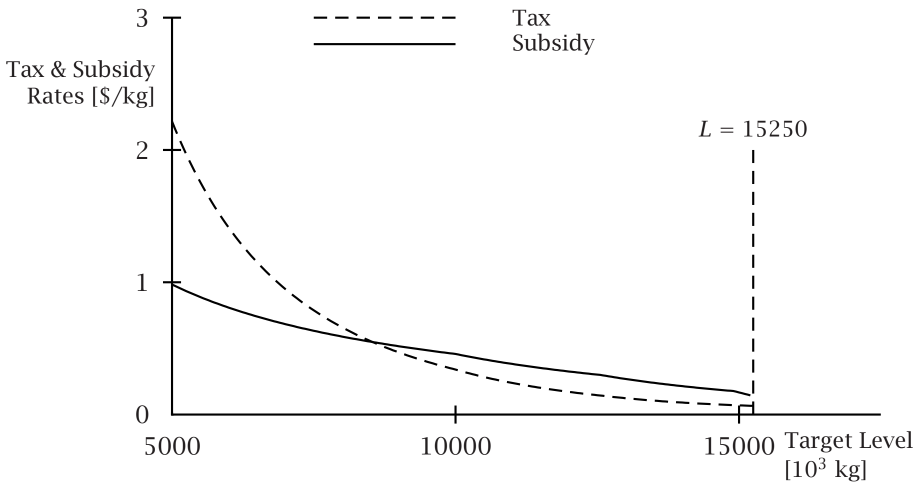

Figure 14.4: Optimal tax and subsidy rates

The curves in Figure 14.5 indicate that the companies with the lowest initial waste concentrations (i.e. companies 1 and 2) have a net expenditure to the government, while the companies with the highest initial waste concentrations (i.e. companies 3 and 4) have a net income from the government. It seems unfair that largest contributors to the overall waste dumped into the river receive the largest subsidies to remove it. On the other hand, their actions have the strongest effect on the amount of waste to be removed through filtering. As shown in the next section, given the status quo, the tax incentive program seems to lead to the most efficient solution for society as a whole.

... on payments to/from the government

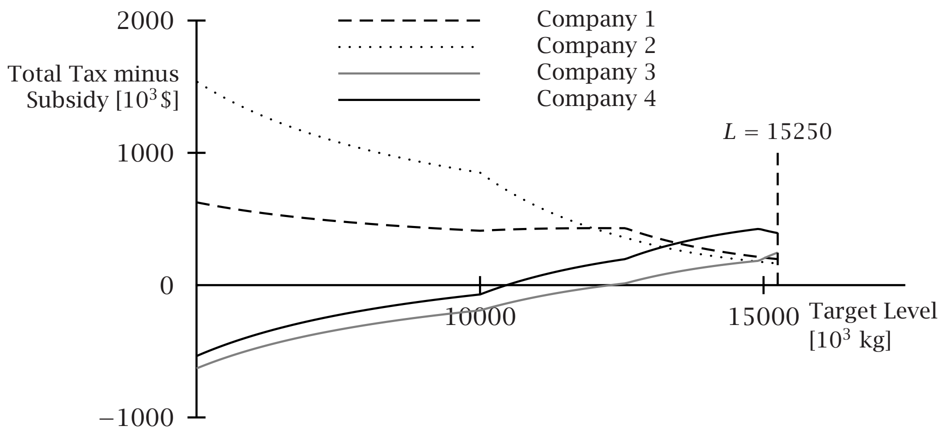

Figure 14.5: The total tax - subsidy per company (i.e. $q_j(T(d_j - x_j) - Sx_j)$)

Observe that for each fixed value of the target level L, the sum of the four function values equals the total annual cost K of government monitoring. For the case when $L \geq 15,250[10^3\text{kg}]$, the subsidy is zero and the aggregated tax paid covers the monitoring cost.

14.5 Alternative solution approach

The solution approach described in Section 14.3 is an intuitive and correct way to solve the two-level model. In this section an alternative single-step solution method is presented. This alternative method is not as general as the previous algorithmic approach, but turns out to be very efficient in this case. The method consists of solving a single auxiliary model, followed by a simple computation to determine the corresponding values of T and S. The proof of correctness is based on examining the underlying optimality conditions.

This section

Optimality policy receiving models

The cost minimization model for each individual manufacturing company j, from Section 14.2, is as follows.

$$\underset{x_j|T,S}{\textbf{Minimize:}} \quad q_j\Big[\big(\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j}\big) + T(d_j - x_j) - Sx_j\Big] \qquad \forall j$$

As previously stated, these optimization models are strictly convex. This implies the existence of a necessary and sufficient optimality condition for each company j. This condition results from setting the first derivative with respect to the variable x_j equal to zero.

$$q_j\Big[\frac{c_j}{(d_j - x_j)^2} - (T + S)\Big] = 0 \qquad \forall j$$

There is a unique solution value x_j for each unique value of $T + S$.

An auxiliary model

Consider the following model in which there is no tax incentive scheme, and all companies work together to attain the goal set out by the government. The total cost of removing waste *for all companies combined* is minimized subject to the restriction that the overall waste production must equal the target level L. This situation is not fully realistic since the companies do not actually collaborate. However, such a model will produce the target waste level at the lowest overall cost.

Minimize:

$$\sum_j q_j\left(\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j}\right)$$

Subject to:

$$\sum_j q_j(d_j - x_j) = L$$

Optimality auxiliary model

This model is a constrained convex minimization model with a unique solution. The necessary and sufficient conditions for optimality are derived from the associated Lagrangian function

$$L(\ldots, x_j, \ldots, \lambda) = \sum_j q_j\left(\frac{c_j}{d_j - x_j} - \frac{c_j}{d_j}\right) - \lambda\Big[\sum_j q_j(d_j - x_j) - L\Big]$$

By setting the first derivatives with respect to the variables x_j (for each j) and λ equal to zero, the following conditions result.

$$\begin{aligned} q_j\Big[\frac{c_j}{(d_j - x_j)^2} + \lambda\Big] &= 0 \qquad \forall j \\ \sum_j q_j(d_j - x_j) &= L \end{aligned}$$

Similarity in optimality conditions

Note that the optimal value of λ is such that all equations are satisfied. In addition, observe that the optimality condition for the policy receiving models and the first set of optimality conditions for the above auxiliary model are similar in structure. By equating the quantities $-(T+S)$ and λ, these optimality conditions become identical.

Optimal tax setting

By solving the auxiliary model and using the value of λ (the shadow price of the waste equality constraint) produced by the nonlinear solution algorithm, the optimal value of $T + S$ is determined. This, together with the known relationship between the values of T and S, give rise to two equations in two unknowns.

$$\begin{aligned} -(T+S) &= \lambda \\ S &= (TL-K)/(G-L) \end{aligned}$$

The solution for T and S can be obtained after simple manipulations.

$$\begin{aligned} T &= (K-\lambda(G-L))/G \\ S &= (-K-\lambda L)/G \end{aligned}$$

Verifying the results

The value of λ obtained from solving the auxiliary model for the example data provided in Section 14.1 is -0.6232. The resulting value of T is 0.239 [$/kg] and the corresponding value of S is 0.384 [$/kg]. These values, when provided as input to the individual minimizing manufacturing companies, produce the same response x_j as the values of x_j obtained by solving the auxiliary model.

14.6 Summary

This chapter presented a worked example of a two-level decision problem. The example was of a government wanting to control the annual amount of waste dumped into a river by setting a tax for waste production and a subsidy for waste removal. The manufacturing companies select their own waste removal level based on cost minimization considerations. A two-level model was developed in detail and two distinct solution procedures were proposed. The first procedure is a bisection algorithm to determine the optimal tax rate for the government. This approach consists of a hunt phase to determine a search interval, followed by a bisection phase to determine the optimal solution. The second solution procedure is based on solving an auxiliary model in which all individual companies collaborate to meet collectively the target waste level set by the government. It was demonstrated that the shadow price on the waste limitation constraint provided enough information to determine the optimal tax rate. Both the optimal solution and the results of sensitivity experiments were reported in detail.

Exercises

14.1 Implement the policy receiving submodel of Section 14.2 using the data provided in Section 14.1, together with a fixed tax rate of 0.25 and a fixed subsidy rate of 0.41. Verify that the solution produced with Aimms coincides with a point along the curve presented in Figure 14.1.

14.2 Implement the algorithm approach described in Section 14.3, and perform the experiments explained in Section 14.4 to study the effects of changing target levels.

14.3 Implement the alternative solution approach described in Section 14.5, and verify whether the results of the two algorithmic approaches coincide.